The Prime Index Graph of a Group

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Abstract

Let G be a group. The prime index graph of G, denoted by $\Pi(G)$, is the graph whose vertex set is the set of all subgroups of G and two distinct comparable vertices H and K are adjacent if and only if the index of H in K or the index of K in H is prime. In this paper, it is shown that for every group G, $\Pi(G)$ is bipartite and the girth of $\Pi(G)$ is contained in the set $\{4, \infty\}$. Also we prove that if G is a finite solvable group, then $\Pi(G)$ is connected.

1 Introduction

Let Γ be a graph. We say that Γ is *connected* if there is a path between any two distinct vertices of Γ. We denote by $d(v)$, the degree of a vertex v in Γ. A graph in which every vertex has the same degree is called a *regular graph*. If all vertices have degree k, then the graph is said to be k*-regular*. The *girth* of Γ, denoted by $gr(\Gamma)$, is the length of a shortest cycle in Γ (We say that $gr(\Gamma) = \infty$ if Γ contains no cycle). A *null graph* is a graph with no edges. A *forest* is a graph with no cycle. We denote the complete graph, the path and the cycle of order n by K_n, P_n and C_n, respectively. We use n*-cycle* to denote the cycle of order n, where $n \geq 3$. The *Cartesian product* of two graphs Γ and Ω is denoted by $\Gamma \Box \Omega$. The *hypercube graph* Q_s is the Cartesian product of s copies of P_2.

Let G be a group. We denote the identity element of G by e. The derived subgroup of G is denoted by G' and $G^{(n+1)} = (G^{(n)})'$, where n is a positive integer. For any subgroup H of G, the intersection of all the conjugates of H in G is denoted by $\mathrm{Core}_G(H)$. Let $x \in G$. Then the subgroup generated by x is denoted by $\langle x \rangle$. As usual, $\mathbb{Z}_n$, A_n and S_n denote the group of integers modulo n, the alternating group and the symmetric group of degree n, respectively. For a fixed prime p, the *quasicyclic* p*-group* is denoted by $\mathbb{Z}(p^\infty)$. Also the projective special linear group of degree n over the field $\mathbb{Z}_p$ is denoted by $\mathrm{PSL}(n, p)$.

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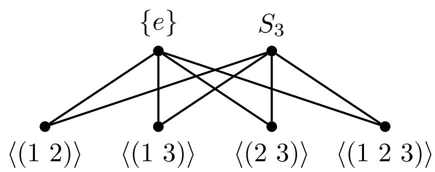
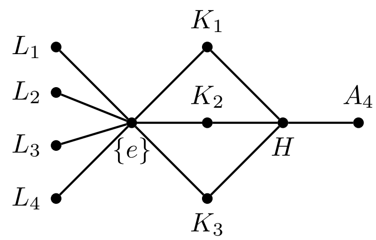
There are several graphs associated with groups, for instance non-commuting graph of a group, intersection graph of subgroups of a group, and subgroup graph of a group. (See [1, 2, 8].) The subgroup graph of a group G is defined as the graph of its lattice of subgroups, that is, the graph whose vertices are the subgroups of G such that two subgroups H and K are adjacent if one of H or K is maximal in the other. In this article, we introduce and investigate the *prime index graph* of G, denoted by $\Pi(G)$. It is an undirected graph whose vertices are all subgroups of G and two distinct comparable vertices H and K are adjacent if and only if $[H : K]$ or $[K : H]$ is prime. Clearly, the prime index graph of G is a subgraph of the subgroup graph of G and whenever G is a nilpotent group, see [12, p.143], then these two graphs are coincide. In follows the prime index graphs of S_3 and A_4 are given. Note that $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $K_i \cong \mathbb{Z}_2$, for $i = 1, 2, 3$ and $L_j \cong \mathbb{Z}_3$, for $j = 1, \ldots, 4$.



$\Pi(S_3)$



$\Pi(A_4)$

Here we show that for every group G, $\Pi(G)$ is a bipartite graph and $gr(\Pi(G)) \in \{4, \infty\}$. We prove that for any finite abelian group G, $\Pi(G)$ is a regular graph if and only if $\Pi(G)$ is a hypercube graph. Finally, we study the connectivity of $\Pi(G)$ and we show that for every finite solvable group G, $\Pi(G)$ is a connected graph. Among other results, we prove that if $\Pi(G)$ is a connected graph and N is a normal subgroup of G, then both graphs $\Pi(N)$ and $\Pi(G/N)$ are connected.

2 The Prime Index Graphs are Bipartite

In this section, we show that the prime index graph of a group G is bipartite. To see this, we prove a stronger result. First we define a directed graph $\overrightarrow{\Gamma}(G)$. It is a directed graph whose vertex set is the set of all subgroups of G and for every two distinct vertices H and K, there is an arc from H to K, whenever $H \subseteq K$ and $[K : H] = r$, for some positive integer r. Suppose that r is the weight of the arc from H to K.

Theorem 1. *Let C be a cycle of $\overrightarrow{\Gamma}(G)$. Then the product of weights of all clockwise arcs of C is equal to the product of weights of all counter-clockwise arcs of C.*

Proof. Let C be a cycle of $\overrightarrow{\Gamma}(G)$ of length n. We prove the theorem by induction on n. Clearly, for $n = 3$ the assertion holds. Now, suppose that $n > 3$ and the assertion is true for every integer m,

$3 \le m < n$. If C contains a directed path P of length 2, such as $H \xrightarrow{r} K \xrightarrow{s} L$, then we replace P with the path $H \xrightarrow{rs} L$. Hence by the induction hypothesis the result holds. Otherwise, C contains a path of the form $H \xrightarrow{r} K \xleftarrow{s} L \xrightarrow{t} M$. We consider two cases:

Case 1. If $H \cap L$ is not a vertex of C, then we replace $H \xrightarrow{r} K \xleftarrow{s} L$ with the path $H \xleftarrow{s'} H \cap L \xrightarrow{r'} L$, where $[L : H \cap L] = r'$ and $[H : H \cap L] = s'$. Note that $s'r = r's$ and so $r/s = r'/s'$. Next, we replace $H \cap L \xrightarrow{r'} L \xrightarrow{t} M$ with $H \cap L \xrightarrow{r't} M$. Thus we find a cycle C_1 of length $n-1$ and by the induction hypothesis $r'ta = s'b$, where a is the product of weights of all clockwise arcs of C_1 except the weight of $H \cap L \xrightarrow{r't} M$ and b is the product of weights of all counter-clockwise arcs of C_1 except the weight of $H \xleftarrow{s'} H \cap L$. Hence $r/s = r'/s' = b/ta$ and so $rta = sb$. It is clear that rta is the product of weights of all clockwise arcs of C and sb is the product of weights of all counter-clockwise arcs of C. The result holds.

Case 2. Assume that $H \cap L$ is a vertex of C. Clearly, $H \cap L \ne H$ or $H \cap L \ne L$. With no loss of generality, suppose that $H \cap L \ne H$. By adding the arc $H \cap L \to H$, we find two cycles C_1 and C_2 of lengths less than n. Let $[H : H \cap L] = s'$. Assume that the arc $H \cap L \xrightarrow{s'} H$ is a clockwise arc of C_1. So $H \cap L \xrightarrow{s'} H$ is a counter-clockwise arc of C_2. Now, by the induction hypothesis, $s' = b_1/a_1 = a_2/b_2$, where a_1 is the product of weights of all clockwise arcs of C_1 except the weight of $H \cap L \xrightarrow{s'} H$, a_2 is the product of weights of all clockwise arcs of C_2, b_1 is the product of weights of all counter-clockwise arcs of C_1, and b_2 is the product of weights of all counter-clockwise arcs of C_2 except the weight of $H \cap L \xrightarrow{s'} H$. Thus $a_1 a_2 = b_1 b_2$. The proof is complete. $\square$

Now, we are in a position to prove the following corollary.

Corollary 1. *Let G be a group. Then $\Pi(G)$ is bipartite.*

Proof. We show that every cycle of $\Pi(G)$ is an even cycle. If $\Pi(G)$ has a cycle C, we may assume that C is a cycle in $\overrightarrow{\Gamma}(G)$. Now, by Theorem 1, since all weights are primes, the number of clockwise arcs of C is equal to the number of counter-clockwise arcs of C. Hence C is an even cycle. This implies that $\Pi(G)$ is a bipartite graph. $\square$

If G is a non-trivial group and $e \ne x \in G$, then $\langle x \rangle$ contains a subgroup of prime index and hence $d(\langle x \rangle) \ge 1$. So $\Pi(G)$ is not a null graph.

Lemma 1. *Let G be a group. Then $\Pi(G)$ is a complete bipartite graph if and only if G is a cyclic group of prime order or $|G| = pq$, for some primes p and q.*

Proof. Clearly, if $G \cong \mathbb{Z}_p$, then $\Pi(G) \cong K_2$. Also if $|G| = pq$, then $\Pi(G)$ is a complete bipartite graph whose one part contains all subgroups of G of orders p or q and the other part contains $\{e\}$ and G. Conversely, assume that $\Pi(G)$ is complete bipartite. If $\{e\}$ and G are contained in two different parts of $\Pi(G)$, then $G \cong \mathbb{Z}_p$, where p is a prime number. Otherwise, there exists a subgroup H of G adjacent to both $\{e\}$ and G. Thus $|G| = pq$, for some primes p and q. $\square$

The following theorem shows that if $\Pi(G)$ contains a cycle C, then $gr(\Pi(G)) = 4$.

Theorem 2. *Let G be a group. Then $gr(\Pi(G)) \in \{4, \infty\}$.*

Proof. First assume that G is finite and $|G| = p_1^{n_1} \cdots p_s^{n_s}$, where $p_1, \ldots, p_s$ are distinct primes and $n_1, \ldots, n_s$ are positive integers. Suppose that L_i is a Sylow p_i-subgroup of G, for $i = 1, \ldots, s$. First assume that L_i contains two distinct maximal subgroups H and K, for some i. Since H and K are normal subgroups of L_i, so $HK = L_i$. This implies that $|H \cap K| = p_i^{n_i - 2}$ and hence $L_i - H - H \cap K - K - L_i$ is a 4-cycle in $\Pi(G)$. So by Corollary 1, $gr(\Pi(G)) = 4$. Next, assume that L_i contains a unique maximal subgroup, for $i = 1, \ldots, s$. Hence all Sylow subgroups of G are cyclic. Now, by [10, Theorem 10.26], G is a supersolvable group. If $s \geq 2$, then G has a subgroup K of order $p_1 p_2$ ([10, p.292]). Let H_i be a subgroup of K of order p_i, for $i = 1, 2$. Hence $\{e\} - H_1 - K - H_2 - \{e\}$ is a 4-cycle in $\Pi(G)$ and so by Corollary 1, $gr(\Pi(G)) = 4$. If $s = 1$, then $G \cong \mathbb{Z}_{p_1^{n_1}}$. Thus $\Pi(G) \cong P_{n_1+1}$ and $gr(\Pi(G)) = \infty$.

Now, suppose that G is infinite and $\Pi(G)$ contains a cycle C. It is easy to see that C should contain a path of the form $M - H - N$, where H, M and N are subgroups of G and furthermore M and N are maximal subgroups of H. If both M and N are normal subgroups of H, then $[M : M \cap N] = [MN : N] = [H : N]$ and similarly $[N : M \cap N] = [H : M]$. Thus $H - M - M \cap N - N - H$ is a 4-cycle in $\Pi(G)$. Now, assume that M is not a normal subgroup of H. Then $M - H - xMx^{-1}$ is a path in $\Pi(G)$, for some $x \in G$. Therefore, $M/\mathrm{Core}_H(M) - H/\mathrm{Core}_H(M) - xMx^{-1}/\mathrm{Core}_H(M)$ is a path in $\Pi(H/\mathrm{Core}_H(M))$. Clearly, $H/\mathrm{Core}_H(M)$ is a finite group which is not a cyclic p-group. So by the previous paragraph, $gr(\Pi(H/\mathrm{Core}_H(M))) = 4$ and hence $gr(\Pi(G)) = 4$. $\square$

By the proof of the previous theorem, we have the following corollary.

Corollary 2. *If G is a finite group or an infinite abelian group, then $\Pi(G)$ is a forest if and only if G is isomorphic to either $\mathbb{Z}_{p^n}$ or $\mathbb{Z}(p^\infty)$, where p is a prime and n is a positive integer.*

Proof. Suppose that $\Pi(G)$ is a forest. If G is finite, then by the proof of Theorem 2, $G \cong \mathbb{Z}_{p^n}$, for some prime number p and positive integer n. If G is an infinite abelian group, then G is a torsion p-group. (Note that $gr(\Pi(\mathbb{Z})) = 4$ and if G has two elements of orders p and q, then $\mathbb{Z}_{pq}$ is a subgroup of G, where p, q are distinct primes.) Also by the proof of Theorem 2, every finite subgroup of G is cyclic. Thus G has no non-trivial direct summand. Now, by [9, p.110], $G \cong \mathbb{Z}(p^\infty)$, for some prime p. Clearly, $\Pi(\mathbb{Z}(p^\infty))$ is a disjoint union of an isolated vertex and an infinite path. The proof is complete. $\square$

In the following theorem, we consider the prime index graph of cyclic groups.

Theorem 3. *Let $n = p_1^{n_1} \cdots p_s^{n_s}$, where $p_1, \ldots, p_s$ are distinct primes and $n_1, \ldots, n_s$ are positive integers. Then $\Pi(\mathbb{Z}_n) \cong P_{n_1+1} \square \cdots \square P_{n_s+1}$.*

Proof. We know that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_s^{n_s}}$. If H and K are two distinct subgroups of $\mathbb{Z}_n$, then $H \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ and $K \cong \mathbb{Z}_{p_1^{\beta_1}} \times \cdots \times \mathbb{Z}_{p_s^{\beta_s}}$, where $0 \leq \alpha_i, \beta_i \leq n_i$ for $i = 1, \ldots, s$. So H and K are

adjacent if and only if there exists an integer j, $1 \leq j \leq s$, such that $\alpha_i = \beta_i$ for $i \neq j$ and $\alpha_j = \beta_j \pm 1$. Thus $\Pi(\mathbb{Z}_n) \cong \Pi(\mathbb{Z}_{p_1^{n_1}}) \Box \cdots \Box \Pi(\mathbb{Z}_{p_s^{n_s}})$ and $\Pi(\mathbb{Z}_n) \cong P_{n_1+1} \Box \cdots \Box P_{n_s+1}$. $\Box$

Theorem 4. *Let G be a finite abelian group. If $\Pi(G)$ is regular, then $G \cong \mathbb{Z}_{p_1 \cdots p_s}$ and $\Pi(G) \cong Q_s$, where $p_1, \ldots, p_s$ are distinct prime numbers.*

Proof. Let $|G| = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where $p_1, \ldots, p_s$ are distinct primes and $\alpha_1, \ldots, \alpha_s$ are positive integers. Assume that $G \cong \mathbb{Z}_{p_1^{\alpha_{11}}} \times \cdots \times \mathbb{Z}_{p_1^{\alpha_{1k_1}}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_{s1}}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_{sk_s}}}$, where k_i is a positive integer and $\alpha_{i1} + \cdots + \alpha_{ik_i} = \alpha_i$, for $i = 1, \ldots, s$. We claim that $k_i = 1$ for each i, $1 \leq i \leq s$. By contradiction assume that $k_i \neq 1$, for some i, $1 \leq i \leq s$. Let $n(k_i, p_i)$ be the number of subgroups of order p_i in $\mathbb{Z}_{p_i^{\alpha_{i1}}} \times \cdots \times \mathbb{Z}_{p_i^{\alpha_{ik_i}}}$. Obviously, the number of subgroups of order p_i in two groups $\mathbb{Z}_{p_i^{\alpha_{i1}}} \times \cdots \times \mathbb{Z}_{p_i^{\alpha_{ik_i}}}$ and $(\mathbb{Z}_{p_i})^{k_i}$ are the same. Hence by [3, p.59], we have $n(k_i, p_i) = (p_i^{k_i} - 1)/(p_i - 1)$. Clearly, $d(\mathbb{Z}_{p_i^{\alpha_{i1}}}) = 1 + n(k_i - 1, p_i) + \sum_{j \neq i} n(k_j, p_j)$ and $d(\{e\}) = \sum_{j=1}^{s} n(k_j, p_j)$. Since $\Pi(G)$ is a regular graph, so $n(k_i, p_i) = 1 + n(k_i - 1, p_i)$. This implies that $p_i^{k_i - 1} = 1$ and hence $k_i = 1$, a contradiction. The claim is proved. Thus G is a cyclic group of order $p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and by Theorem 3, $\Pi(G) \cong P_{\alpha_1+1} \Box \cdots \Box P_{\alpha_s+1}$. Now, since $\Pi(G)$ is a regular graph, $\alpha_1 = \cdots = \alpha_s = 1$ and $\Pi(G) \cong Q_s$. $\Box$

Theorem 5. *Let G be a finite group. If $\Pi(G)$ is a 2-regular graph, then $\Pi(G) \cong C_4$ and $G \cong \mathbb{Z}_{pq}$, where p and q are distinct primes.*

Proof. Since $d(\{e\}) = 2$, the order of G has at most two distinct prime divisors. Clearly, a p-group cannot have exactly two subgroups of order p. So assume that p and q are two distinct prime divisors of $|G|$. Suppose that H and K are subgroups of G such that $|H| = p$ and $|K| = q$. Since $d(\{e\}) = 2$, H and K are normal subgroups of G. Hence HK is a subgroup of G and $\{e\} - H - HK - K - \{e\}$ is a cycle in $\Pi(G)$. Now since $\Pi(G)$ is a 2-regular graph, H is a Sylow p-subgroup and K is a Sylow q-subgroup of G. Thus $G = HK \cong \mathbb{Z}_{pq}$ and $\Pi(G) \cong C_4$. $\Box$

3 Connectivity

In this section, we study those groups whose prime index graphs are connected. First we have the following lemma.

Lemma 2. *Let G be an infinite group. Then $\Pi(G)$ is not connected. Moreover, if G is a simple group, then G is an isolated vertex in $\Pi(G)$.*

Proof. It is clear that if G is an infinite group, then there is no path between $\{e\}$ and G in $\Pi(G)$. So $\Pi(G)$ is not connected. If G is an infinite simple group, by [10, Corollary 4.15], G cannot have a proper subgroup of finite index. Hence G is an isolated vertex of $\Pi(G)$. $\Box$

By [10, p.292], a finite group G is supersolvable if and only if each subgroup of G satisfies the converse of Lagrange's Theorem. So for finite supersolvable groups G such as finite abelian groups and finite p-groups, $\Pi(G)$ is connected. (Note that every subgroup of G is connected to $\{e\}$.)

Theorem 6. *Let G be a finite group and N be a normal subgroup of G. If $\Pi(N)$ is a connected graph and also for every subgroup H/N of G/N, $\Pi(H/N)$ is a connected graph, then $\Pi(G)$ is connected.*

Proof. Assume that H is a subgroup of G. Hence $\Pi(HN/N)$ is a connected graph. Since $HN/N \cong H/(H \cap N)$, so the graph $\Pi(H/H \cap N)$ is connected. This implies that there is a path between H and $H \cap N$ in $\Pi(G)$. Now, since $\Pi(N)$ is connected, there is a path between $H \cap N$ and $\{e\}$ in $\Pi(N)$. Thus every subgroup of G is connected to $\{e\}$. Therefore $\Pi(G)$ is connected. $\square$

Now, we prove that the prime index graph of every finite solvable group is connected.

Theorem 7. *Let G be a finite solvable group. Then $\Pi(G)$ is connected.*

Proof. Since G is a solvable group, $G^{(n)} = \{e\}$, for some positive integer n. We prove the theorem by applying the induction on n. If $G' = \{e\}$, then G is an abelian group and so $\Pi(G)$ is a connected graph. Assume that $n > 1$ and $G^{(n)} = \{e\}$. By the induction hypothesis, $\Pi(G')$ is connected. Now, by Theorem 6, $\Pi(G)$ is a connected graph. $\square$

If G is a group of odd order, then G is solvable (Feit-Thompson Theorem [5]) and by Theorem 7, $\Pi(G)$ is connected. Moreover, suppose that $|G| = 2^n m$, where m and n are positive integers with m odd. If G has a cyclic Sylow 2-subgroup, then by [4, p.148], G has a normal subgroup of order m and hence G is a solvable group. Thus $\Pi(G)$ is a connected graph. Since every subgroup of a solvable group is solvable, by Theorems 6 and 7, we have the next result.

Corollary 3. *Let G be a finite group and N be a normal subgroup of G. If $\Pi(N)$ is a connected graph and G/N is a solvable group, then $\Pi(G)$ is connected.*

Theorem 8. *Let G be a group and N be a normal subgroup of G. If $\Pi(G)$ is a connected graph, then $\Pi(N)$ and $\Pi(G/N)$ are connected graphs.*

Proof. First we prove that $\Pi(N)$ is a connected graph. Let H and K be two distinct subgroups of N. Since $\Pi(G)$ is a connected graph, so there is a path $H - L_1 - \cdots - L_t - K$ from H to K in $\Pi(G)$. We claim that by removing the same consecutive vertices in $H - L_1 \cap N - \cdots - L_t \cap N - K$ and keeping one of them we obtain a walk from H to K in $\Pi(N)$. With no loss of generality, assume that $L_i \subseteq L_{i+1}$ and $[L_{i+1} : L_i] = p$, for some prime number p. Thus $L_i \cap N \subseteq L_{i+1} \cap N$ and we have

$$[L_{i+1} \cap N : L_i \cap N] = \frac{|L_{i+1} \cap N|}{|L_i \cap N|} = \frac{|L_i N|}{|L_{i+1} N|} \frac{|L_{i+1}|}{|L_i|}.$$

Hence $[L_{i+1}N : L_iN][L_{i+1} \cap N : L_i \cap N] = p$. Therefore $L_{i+1} \cap N = L_i \cap N$ or $[L_{i+1} \cap N : L_i \cap N] = p$. So the claim is proved. Hence there is a path from H to K in $\Pi(N)$ which implies that $\Pi(N)$ is connected.

Next, assume that H and K are two distinct subgroups of G containing N. Suppose that $H - L_1 - \cdots - L_t - K$ is a path from H to K in $\Pi(G)$. Similar to the previous case, one can prove that $H/N - L_1N/N - \cdots - L_tN/N - K/N$ is a walk from H/N to K/N in $\Pi(G/N)$. Thus $\Pi(G/N)$ is also a connected graph. $\square$

Now, we propose the following problem.

Problem. Let G be a group and N be a normal subgroup of G. If $\Pi(N)$ and $\Pi(G/N)$ are both connected, then is it true that $\Pi(G)$ is connected?

By Theorem 8, we have the next corollary.

Corollary 4. *Let $G \cong H \times K$, for some groups H and K. If $\Pi(G)$ is connected, then both $\Pi(H)$ and $\Pi(K)$ are connected.*

We close this article by the study of the connectivity of $\Pi(A_n)$ and $\Pi(S_n)$. Moreover, we show that the prime index graph of all groups up to 500 elements is connected except for A_6.

Remark. Let n be a positive integer. Then $\Pi(A_n)$ is connected if and only if $n \leq 5$. Also, $\Pi(S_n)$ is a connected graph if and only if $n \leq 5$. To prove the remark first assume that $n \leq 4$. Hence A_n is a solvable group and by Theorem 7, $\Pi(A_n)$ is a connected graph. If $n = 5$, we know that every proper subgroup of A_5 is solvable and A_5 contains a maximal subgroup of prime index, then $\Pi(A_5)$ is connected. Also if $n \leq 5$, since A_n is a normal subgroup of S_n and $\Pi(A_n)$ is connected, by Corollary 3, $\Pi(S_n)$ is connected. Now, assume that $n > 5$. If n is not a prime number, then by [6, p.305], A_n has no subgroup of prime index and hence A_n is an isolated vertex of $\Pi(A_n)$. Otherwise, if H is a maximal subgroup of A_n of prime index, then $H \cong A_{n-1}$ (see [6, p.305]). Since $n - 1$ is not a prime number, so $\Pi(A_n)$ is not connected. Thus by Theorem 8, $\Pi(S_n)$ is not connected.

Theorem 9. *Let G be a group and $|G| \leq 500$. If $\Pi(G)$ is not connected, then $G \cong A_6$.*

Proof. Suppose that G is the smallest group such that $\Pi(G)$ is not a connected graph. By Theorem 6, one can see that G is a simple group. Note that by the remark, $\Pi(A_6)$ is not a connected graph. On the other hand by [11, p.295], if G is a non-abelian simple group of order at most 500, then G is isomorphic to one of the groups A_5, $\mathrm{PSL}(2,7)$, or A_6. By remark, $\Pi(A_5)$ is connected. Also by [13, Theorem 6.26], $\mathrm{PSL}(2,7)$ contains a maximal subgroup of index 7 and by [7, p.191], all subgroups of $\mathrm{PSL}(2,7)$ are solvable. Hence $\Pi(\mathrm{PSL}(2,7))$ is connected. Thus $G \cong A_6$. Finally, for every non-abelian group G with $360 < |G| \leq 500$, since G is not a simple group, so G has a non-trivial proper normal subgroup N. Clearly, $|N|$ and $|G/N|$ are both less than 360. Thus by Theorem 6, $\Pi(G)$ is a connected graph. $\square$

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